Parameterization conversions for some von Bertalanffy growth curves

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Abstract

The goal of this working paper is to show the one-to-one relationships that exist between three different parameterizations of the von Bertalanffy growth curve. The three forms of the von Bertalanffy curve considered here are the (i) standard parameterization, (ii) the reference-age parameterization, and the (iii) Stock Synthesis parameterization. The one-to-one relationships between the parameters of three forms of the 3-parameter von Bertalanffy curve are derived. These one-to-one mappings provide a direct conversion of the parameters of one form of the von Bertalanffy curve to another for stock assessment applications.

Introduction

The conversion of parameters among alternative forms of growth curves may be needed for stock assessment applications. In this paper, we show the one-to-one relationships among parameters for three forms of the von Bertalanffy growth curve: the standard form, the modified or reference-age form, and the Stock Synthesis modified form. These one-to-one relationships provide a unique conversion of the parameters of one form of the growth curve into another form for stock assessment applications. In particular, the one-to-one relationship between the parameters of the standard, modified or reference-age, and Stock Synthesis forms of the von Bertalanffy curve are expressed for application in the 2023 Western and Central North Pacific Ocean striped marlin stock assessment conducted by the ISC Billfish Working Group.

Methods

We begin by providing parameter definitions and notation for the standard, modified and Stock Synthesis forms of the von Bertalanffy growth curve. We then describe the resulting one-to-one relationships between the parameters of the three forms of the growth curve for stock assessment applications.

Standard von Bertalanffy Curve

The standard parameterization of the von Bertalanffy growth curve to predict mean length (L) as a function of age (a) is

(1) $$L(a) = L_{\inf}\left(1 - \exp\left[-k\left(a - a_0\right)\right]\right)$$

Where the three parameters of the standard von Bertalanffy function are $\underline{\theta}_{VB} = \left(L_{\inf}, k, a_0\right)$ and

- $L_{\inf}$ is the asymptotic mean length as age becomes unbounded and approaches infinity
- k is the growth rate coefficient, or Brody growth coefficient
- a_0 is the age at which the predicted mean length is zero $L\left(a_0\right) = 0$

The three parameters of the standard von Bertalanffy curve $\underline{\theta}_{VB} = \left(L_{\inf}, k, a_0\right)$ satisfy the following constraints: $L_{\inf} > 0$ and $k > 0$

Modified von Bertalanffy Curve

The modified (or reference-age) parameterization of the von Bertalanffy growth curve to predict mean length (L) as a function of age (a) is

(2)

$$L(a) = L_{\min} + (L_{\max} - L_{\min}) \frac{1 - \exp\left[-k(a - a_{\min})\right]}{1 - \exp\left[-k(a_{\max} - a_{\min})\right]} = L_{\min} + (L_{\max} - L_{\min}) \frac{1 - \exp\left[\log(C) \cdot (a - a_{\min})\right]}{1 - \exp\left[\log(C) \cdot (a_{\max} - a_{\min})\right]}$$

Where the three parameters of the modified von Bertalanffy function are $\underline{\theta}_{MVB} = (L_{\min}, L_{\max}, k)$ or $\underline{\theta}_{MVB} = (L_{\min}, L_{\max}, C)$ where $C = \exp[-k]$ and

- $L_{\min}$ is the mean length at the minimum reference age $a_{\min}$
- $L_{\max}$ is the mean length at the maximum reference age $a_{\max}$
- k is the growth rate coefficient, or $C = \exp[-k]$ is the log-scale growth coefficient

Note that in some applications, the Brody growth rate coefficient is expressed on log-scale as $\log(C) = -k$ for parameter estimation purposes (Brodziak and Mikus 2000, Andrews et al. 2020), where $C > 0$ and $C^{(a - a_{\min})} = \exp\left[\log(C) \cdot (a - a_{\min})\right] = \exp\left[-k(a - a_{\min})\right]$. The three parameters of the modified von Bertalanffy curve $\underline{\theta}_{MVB} = (L_{\min}, L_{\max}, k)$ or $\underline{\theta}_{MVB} = (L_{\min}, L_{\max}, C)$ satisfy the following constraints: $L_{\max} > L_{\min} > 0$ and $k > 0$ and $a_{\max} > a_{\min} > 0$ and $C > 0$

Stock Synthesis form of von Bertalanffy Curve

The Stock Synthesis parameterization of the von Bertalanffy growth curve to predict mean length (L) as a function of age (a) is

(3) $$L(a) = L_{\inf} + (L_1 - L_{\inf}) \exp\left[-k(a - a_1)\right]$$

Where the three parameters of the Stock Synthesis von Bertalanffy function are $\underline{\theta}_{SVB} = (L_{\inf}, L_1, k)$ and

- $L_{\inf}$ is the asymptotic mean length as age becomes unbounded and approaches infinity
- L_1 is the mean length at the lower reference age a_1, where $L(a_1) = L_1$ and noting that the lower reference age is often set to be age-1, $a_1 = 1$
- k is the growth rate coefficient, or Brody growth coefficient

The three parameters of the Stock Synthesis von Bertalanffy curve $\underline{\theta}_{SVB} = (L_{\inf}, L_1, k)$ satisfy the following constraints: $L_{\inf} > L_1 > 0$ and $k > 0$ and $a_1 > 0$

Results

Relationships for the Standard and Modified von Bertalanffy Curves

To relate the parameters of the three von Bertalanffy curves, we define a scaling coefficient S as

(4) $$S = \exp\left[-k\left(a_{\max} - a_{\min}\right)\right] = \exp\left[\log(C)\cdot\left(a_{\max} - a_{\min}\right)\right]$$

Given the definition of the scaling coefficient *S*, the parameters for the standard von Bertalanffy function $\underline{\theta}_{VB} = \left(L_{\inf}, k, a_0\right)$ can be expressed in terms of the parameters of the modified function $\underline{\theta}_{MVB} = \left(L_{\min}, L_{\max}, k\right)$ for the asymptotic length as

(5) $$L_{\inf} = \frac{L_{\max} - L_{\min}\cdot S}{1 - S}$$

In addition, for the Brody growth coefficient if it is expressed in log-scale in the modified function as

(6) $$k = -\log(C)$$

And for the age at which the predicted mean length is zero $L\left(a_0\right) = 0$ as

(7) $$a_0 = a_{\min} - \frac{\log\left(\frac{L_{\max} - L_{\min}}{L_{\max} - L_{\min}\cdot S}\right)}{\log(C)}$$

Similarly, the mean lengths at the minimum and maximum reference ages can be expressed in terms of the standard von Bertalanffy parameters as

(8) $$L_{\min} = L_{\inf}\left(1 - \exp\left[-k\left(a_{\min} - a_0\right)\right]\right)$$

And

(9) $$L_{\max} = L_{\inf}\left(1 - \exp\left[-k\left(a_{\max} - a_0\right)\right]\right)$$

Relationships for the Stock Synthesis and Modified von Bertalanffy Curves

The parameters for the Stock Synthesis von Bertalanffy function $\underline{\theta}_{SVB} = \left(L_{\inf}, L_1, k\right)$ can be expressed in terms of the parameters of the modified function $\underline{\theta}_{MVB} = \left(L_{\min}, L_{\max}, k\right)$ for the asymptotic length as

(10) $$L_{\inf} = L_{\min} + \frac{L_{\max} - L_{\min}}{1 - \exp\left[-k\left(a_{\max} - a_{\min}\right)\right]} = L_{\min} + \frac{L_{\max} - L_{\min}}{1 - S} = \frac{L_{\max} - L_{\min}\cdot S}{1 - S}$$

And for the mean length $L\left(a_1\right) = L_1$ at the lower reference age a_1 as

(11) $L_1 = L_{\min}$, where the minimum reference age $a_{\min}$ can be set to be the same age as the lower reference age $a_{\min} = a_1$, or vice versa for $L_{\min} = L_1$

And for the mean length $L_{\max}$ at the maximum reference age in terms of $\underline{\theta}_{SVB} = \left(L_{\inf}, L_1, k\right)$ as

(12) $$L_{\max} = L_1 \cdot \exp\left[-k\left(a_{\max} - a_1\right)\right] + L_{\inf} \cdot \left(1 - \exp\left[-k\left(a_{\max} - a_1\right)\right]\right)$$

Last, we note that the Brody growth coefficient k is the same for both curves.

Relationships for the Stock Synthesis and standard von Bertalanffy Curves

The parameters for the Stock Synthesis von Bertalanffy function $\underline{\theta}_{SVB} = \left(L_{\inf}, L_1, k\right)$ can be expressed in terms of the parameters of the standard function $\underline{\theta}_{VB} = \left(L_{\inf}, k, a_0\right)$ for the mean length $L\left(a_1\right) = L_1$ at the reference age a_1 as

(13) $$L_1 = L_{\inf}\left(1 - \exp\left[-k\left(a_1 - a_0\right)\right]\right)$$

Moreover, for the lower reference age at which the predicted mean length is L_1 as

(14) $$a_0 = a_1 + \frac{\log\left(1 - \frac{L_1}{L_{\inf}}\right)}{k}$$

Similarly, for the age at which the predicted mean length is zero $L\left(a_0\right) = 0$ as

(15) $$a_1 = a_0 - \frac{\log\left(1 - \frac{L_1}{L_{\inf}}\right)}{k}$$

Last, we note that the asymptotic length $L_{\inf}$ and the Brody growth coefficient k are the same for both curves.

Summary

In summary, the working paper provides the one-to-one relationships among parameters for three forms of the von Bertalanffy growth curve: the standard form, the modified or reference-age form, and the Stock Synthesis modified form. These one-to-one relationships provide a unique conversion of the parameters of one form of the growth curve into another form for stock assessment applications. In particular, the one-to-one relationship between the parameters of the standard, modified or reference-age, and Stock Synthesis forms of the von Bertalanffy curve are expressed for application in the 2023 Western and Central North Pacific Ocean striped marlin stock assessment conducted by the ISC Billfish Working Group. In particular, the relationships between the parameters of the standard and modified von Bertalanffy curves are expressed in equations (4) to (9). Equations (10) to (12) provide the parameter conversions between the Stock Synthesis and the modified von Bertalanffy curves. Last, the relationships between the standard and Stock Synthesis forms of the von Bertalanffy curves are given in equations (13) to (15).

References

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