Conversion of Richards Growth Curve Parameters for Western and Central North Pacific Ocean Striped Marlin

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Abstract

The goal of this working paper is to show how to convert the parameters of the best-fitting 4-parameter Richards growth curve used in the Sun et al. (2011) ISC Billfish WG working paper on striped marlin to the parameters of the Richards curve used in the SS3 assessment software. The working paper begins with a description of the asymptotic form of the Richards curve and parameter estimates from Sun et al. (2011). The parameters of the asymptotic Richards curve are then converted to those of the modified form used in the SS3 software where the modified form includes length at minimum and maximum length at reference age parameters, in comparison to an asymptotic length parameter in the asymptotic form. We convert the parameters based on direct calculation of the minimum and maximum length at reference age parameters. We then show that the converted parameters are numerical solutions of necessary conditions for the modified Richards growth curve using Newton's method and provide a summary of the working paper results.

Introduction

The conversion of the 4-parameter asymptotic Richards growth curve to the modified form of the Richards growth curve used in the Stock Synthesis 3 (SS3) software is the goal of this working paper. We begin with a description of the asymptotic Richards growth curve used in Sun et al. (2011). We then show how the parameters of the modified curve can be calculated from the parameters of the asymptotic curve. We then show that the parameters of the modified curve solve the necessary numerical conditions to verify the conversion process and summarize the results.

Methods

The asymptotic form of the Richards growth curve used by Sun et al. (2011) has four parameters to be estimated from observational data. These four parameters are elements of the parameter vector $\underline{\theta}$ for the asymptotic form where $\underline{\theta}=\left(a_0, L_{\inf}, c, m\right)$. Here the length of a fish of age-a is denoted by $L\left(a\right)$. The four parameters of the asymptotic form of the Richards curve are:

- a_0 is the age at which the predicted mean length is zero $L\left(a_0\right)=0$
- $L_{\inf}$ is the asymptotic mean length as age approaches infinity
- c is the growth rate coefficient
- m is the asymptotic curvature parameter

The equation for the asymptotic form of the Richards growth curve model for predicting mean fish length at age-a $L\left(a\right)$ is

$$L\left(a\right)=L_{\inf}\left\{1-\exp\left[-c\left(1-m\right)\left(a-a_0\right)\right]\right\}^{\frac{1}{1-m}} \tag{1}$$

The constraints on the parameters of the asymptotic Richards growth curve are:

(2) $$L(a_0)=0 \text{ and } \lim_{a\to\infty} L(a)=L_{\inf} \text{ and } c>0 \text{ and } m\neq 1$$

These four constraints require that the predicted length at age-a_0 is zero cm, the limit as age becomes unbounded is the asymptotic length $L_{\inf}$, the growth rate coefficient c is always positive and the asymptotic curvature parameter m is not equal to 1.

Maximum likelihood point estimates of the growth parameters of the asymptotic Richards curve for WCNPO striped marlin were calculated by Sun et al (2011, Table 5b) and these estimates are:

(3) $$a_0=-0.40 \text{ and } L_{\inf}=228.7\,cm\,EFL \text{ and } c=0.04 \text{ and } m=-2.05$$

Here the units of fish length have been converted from lower-jaw fork length to eye-fork length based on the length conversion relationship developed in Sun et al. (2011). The next step is to convert the parameters of the asymptotic form to the modified form of the Richards growth curve.

The modified form of the Richards growth curve used in SS3[1] has four parameters to be estimated from observational data. These are the elements of the parameter vector $\underline{\theta}^*$ for the modified form where $\underline{\theta}^*=(L_{\min},L_{\max},k,\beta)$, where $L(a)$ is the mean length of an age-a fish in units of *EFL*. The four parameters of the modified form of the Richards curve are:

- $L_{\min}$ is the mean length at the minimum reference age $a_{\min}$
- $L_{\max}$ is the mean length at the maximum reference age $a_{\max}$
- k is the growth rate coefficient
- β is the curvature parameter

The equation for the modified form of the Richards growth curve model (e.g., Brodziak and Macy 1996, Eqn 1) for predicting mean fish length at age-a $L(a)$ is

(4) $$L(a)=\left\{L_{\min}^{\beta}+\left(L_{\max}^{\beta}-L_{\min}^{\beta}\right)\frac{1-\exp\left[-k\left(a-a_{\min}\right)\right]}{1-\exp\left[-k\left(a_{\max}-a_{\min}\right)\right]}\right\}^{\frac{1}{\beta}}$$

Here we note that the modified Richards curve is equivalent to a von Bertalanffy curve when the curvature parameter is equal to unity, $\beta=1$ (i.e., Schnute 1981). As a result, we note that the growth coefficient k is equivalent to the Brody growth coefficient when $\beta=1$. The minimum

[1]The SS3 software version 3.30.20 is available at https://github.com/nmfs-stock-synthesis/stock-synthesis/releases/tag/v3.30.20 and the code to implement the Richards growth function is listed in the tpl function SS_biofxn.tpl

and maximum reference age settings for the application of the growth curve for the ISC Billfish Working Group's work on the 2023 WCNPO striped marlin stock assessment are: $a_{\min}=0.5\ yr$ and $a_{\min}=15.0\ yr$, respectively. In what follows, the parameters of the

asymptotic Richards curve are converted to the parameters of the modified form of the Richards curve and the conversion results are numerically verified.

Results

The parameters of the asymptotic Richards curve are a function of those of the modified curve by through prediction of the mean length at age at the minimum and maximum reference ages based on the asymptotic growth curve parameters. First, we note that, the relationships between growth parameters of the modified and asymptotic Richards models for k and β are:

(5) $$k = c \cdot \beta \text{ and } \beta = 1 - m$$

Plugging in the parameter estimates of the asymptotic growth curve estimated by Sun et al. (2011) gives the values for the curvature and growth coefficient parameters of the modified Richards curve as

(6) $$\beta = 1 - m = 1 - (-2.05) = 3.05 \text{ and } k = c \cdot \beta = (0.04) \cdot (3.05) = 0.122$$

As a result, the mean length at the minimum reference age of $a_{\min} = 0.5\ yr$ using the asymptotic Richards growth curve gives a minimum reference length of $L_{\min} = L(a_{\min})$ in cm of EFL as

(7) $$L_{\min} = L(0.5) = 228.7\left(1 - \exp\left[-0.122\left(0.5 - (-0.4)\right)\right]\right)^{\frac{1}{3.05}} = 108.869$$

Similarly, the value of the mean length at the maximum reference age of $L_{\max} = L(a_{\max})$ based the parameters from Sun et al. (2011) with $a_{\max} = 15$ years is

(8) $$L_{\max} = L(15) = 228.7\left(1 - \exp\left[-0.122\left(15.0 - (0.4)\right)\right]\right)^{\frac{1}{3.05}} = 216.572$$

Therefore, the parameters of the modified Richards curve are

(9) $$L_{\min} = 108.869,\ L_{\max} = 216.572,\ k = 0.122,\ \beta = 3.05$$

We verify that the parameters of the modified curve in (9) above match those of the asymptotic Richards curve and show that the fitted parameters satisfy the requirement that the predicted mean length at age a_0 is $L(a_0) = 0$. To begin, we note that the requirement that $L(a_0) = 0$ implies that

(10) $$L(a_0) = \left\{ L_{\min}^{\beta} + \left(L_{\max}^{\beta} - L_{\min}^{\beta}\right) \frac{1 - \exp\left[-k\left(a_0 - a_{\min}\right)\right]}{1 - \exp\left[-k\left(a_{\max} - a_{\min}\right)\right]} \right\}^{\frac{1}{\beta}} = 0$$

Exponentiating equation (10) to the power β and substituting for $a_0, a_{\min}, a_{\max}$ and rearranging terms gives the nonlinear function $g(k) = 0$ where

(11) $$g(k)=\exp[-15.4k]-\exp[-0.9k]\cdot\left(\frac{L_{\max}}{L_{\min}}\right)^{\beta}+\left(\frac{L_{\max}}{L_{\min}}\right)^{\beta}-1=0$$

We show that $k=0.122$ is the numerical solution of $g(k)=0$. To do this, we use Newton's method where the $(n+1)^{st}$ estimate of k is given by $k_{n+1}=k_n-\frac{g(k_n)}{g'(k_n)}$ and the derivative of $g(k)$ is $g'(k)=-15.4\exp[-15.4k]+0.9\exp[-0.9k]\left(\frac{L_{\max}}{L_{\min}}\right)^{\beta}$. To start the Newton iterations we set the initial value k_0 to be $k_0=\left(0.9\left(\frac{L_{\max}}{L_{\min}}\right)^{\beta}\right)^{-1}\approx 0.136$, noting that $\exp[-15.4k]\approx 0$. The numerical solution of $g(k)=0$ converged after three iterations and this showed that $k=0.122$ as required.

Table 1. Results of numerically solving the equation $g(k)=0$ to show that $k=0.122$ satisfies the condition that $L(a_0)=0$ for the asymptotic Richard growth curve.

Solving g(k)=0 using Newton's Method					
Iteration number	k(n+1)	k(n)	g(k(n))	g'(k(n))	|k(n+1)-k(n)|
0	0.122571	0.136375	0.063501	4.600257	0.013804
1	0.122001	0.122571	0.002415	4.234730	0.000570
2	0.122000	0.122001	0.000005	4.217530	0.000001
3	0.122000	0.122000	0.000000	4.217495	0.000000

Richards growth curve parameters	
Lmin =	108.86947
Lmax =	216.57176
Lmax/Lmin =	1.98928
B =	3.05
(Lmax/Lmin)^B =	8.14745
Initial k(0) =	0.13638
Calculated value of k =	0.12200

Summary

In summary, the fitted parameters of the modified Richards growth curve for WCNPO striped marlin are $\hat{\underline{\theta}} = \left(\hat{L}_{\min}, \hat{L}_{\max}, \hat{k}, \hat{\beta}\right)$ where

$\hat{L}_{\min} = 108.869,\ \hat{L}_{\max} = 216.572,\ \hat{k} = 0.122,\ \hat{\beta} = 3.05$

Substituting these parameter values into the modified Richards curve (see Eqn 4) for WCNPO striped marlin gives the predicted mean length at age-*a* as

(12)

$$L(a) = \left\{108.869^{3.05} + \frac{\left(216.572^{3.05} - 108.869^{3.05}\right)}{1 - \exp\left[-1.769\right]}\left(1 - \exp\left[-0.122\left(a - 0.5\right)\right]\right)\right\}^{\frac{1}{3.05}}$$

References

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